MATH2230A

Tutorial 3 28-9-2016

Topics: ① Harmonic conjugate
② Exponential function and Logarithmic function
③ Complex exponents
④ Inverse of trigonometric function.

Harmonic Conjugate

Last time:
- A function $f:\mathbb{R}^2 \to \mathbb{R}$ is harmonic if it is twice continously differentiable and $\Delta f = f_{xx} + f_{yy} = 0$.
- If $f = u+iv$ is analytic, then u & v are harmonic.
- Furthermore, u determines v up to an additive constant.

Defn: Given a twice continuously differentiable $u:\mathbb{R}^2 \to \mathbb{R}$.
A harmonic conjugate of u is a twice continuously differentiable function $v:\mathbb{R}^2 \to \mathbb{R}$ s.t. $f = u+iv$ is analytic.
Equivalently, it means u and v satisfy the CR-eqtn:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Q: Given u. How to find v?

Example: 1) Let $u(x,y) = 2x - x^3 + 3xy^2$.
Let v be a harmonic conjugate of u.
Then we have

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$\Rightarrow \begin{cases} v_y = 2 - 3x^2 + 3y^2 & \text{—①} \\ v_x = -6xy & \text{—②} \end{cases}$$

From ②,
$$v_x = -6xy$$
$$\Rightarrow v = -3x^2y + C(y)$$
$$\Rightarrow v_y = -3x^2 + C'(y)$$

From ①, we have $C'(y) = 2 + 3y^2$, $C(y) = 2y + y^3 + C$.

Hence $v = -3x^2y + 2y + y^3 + C$.

Remark: 1) v is a harmonic conjugate of u
$\Leftrightarrow f = u + iv$ is analytic
$\Leftrightarrow -if = v - iu$ is analytic
$\Leftrightarrow (-u)$ is a harmonic conjugate of v.
2) Given v, we can use the same method to find u s.t. $f = u + iv$ is analytic.

Exponential function

Defn: $e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

Example: 1) Show that $|\exp(z^2)| \leq \exp(|z|^2)$.
Ans: $L.H.S. = |\exp(z^2)| = |\exp(x^2 - y^2 + i(2xy))| = e^{x^2-y^2}$
$R.H.S. = \exp(|z|^2) = e^{x^2+y^2}$
$\Rightarrow |\exp(z^2)| \leq \exp(|z|^2)$
Moreover, equality holds iff. $y^2 = 0$, i.e. $z \in \mathbb{R}$.

Now we want to solve the equation $e^z = w$, $w \in \mathbb{C}$.
Write $z = x + iy$, we have

$$e^x \cdot e^{iy} = |w| e^{i\arg(w)}$$

$\Rightarrow \quad e^x = |w| \quad$ and $\quad y = \arg(w)$
$\Rightarrow \quad x = \ln|w| \quad$ and $\quad y = \mathrm{Arg}(w) + 2n\pi$, $n \in \mathbb{Z}$

Logarithmic function

Defn:
- $\log(w) = \ln|w| + i\arg(w)$. (multi-valued function)
- $\mathrm{Log}(w) = \ln|w| + i\,\mathrm{Arg}(w)$. (single-valued function)
- $\log_\alpha(w) = \ln|w| + i\,\mathrm{Arg}_\alpha(w)$, (single-valued function)
  $\mathrm{Arg}_\alpha(w) \in (\alpha, \alpha + 2\pi]$

Example: 1) Find $\log(-1+\sqrt{3}i)$, $\mathrm{Log}(-1+\sqrt{3}i)$ and $\log_\pi(-1+\sqrt{3}i)$

Ans: $\log(-1+\sqrt{3}i) = \ln|-1+\sqrt{3}i| + i\arg(-1+\sqrt{3}i)$
$= \ln 2 + i\left(\frac{2\pi}{3} + 2n\pi\right)$

$\mathrm{Log}(-1+\sqrt{3}i) = \ln 2 + i\left(\frac{2\pi}{3}\right)$

$\log_{\pi}(-1+\sqrt{3}i) = \ln 2 + i\left(\frac{2\pi}{3} + 2\pi\right)$

Complex Exponent

Defn: For $z \neq 0$, $c \in \mathbb{C}$, $z^c = e^{c\log z}$ (multi-valued function)

Example: 1) $(1-i)^{1+i} = e^{(1+i)\log(1-i)}$

$= e^{(1+i)(\ln|1-i| + i\arg(1-i))}$

$= e^{(1+i)(\ln|1-i| + i(-\frac{\pi}{4}+2n\pi))}$

$= e^{(\ln|1-i| + \frac{\pi}{4} - 2n\pi) + i(\ln|1-i| - \frac{\pi}{4} + 2n\pi)}$

$= e^{\ln\sqrt{2} + \frac{\pi}{4} - 2n\pi + i(\ln\sqrt{2} - \frac{\pi}{4})}$

$= \sqrt{2}\, e^{\frac{\pi}{4} - 2n\pi + i(\ln\sqrt{2} - \frac{\pi}{4})}$

2) $(-3)^{-1+\sqrt{3}i} = e^{(-1+\sqrt{3}i)(\ln|-3| + i\arg(-3))}$

$= e^{(-1+\sqrt{3}i)(\ln 3 + i(\pi + 2n\pi))}$

$= e^{-\ln 3 - \sqrt{3}(\pi+2n\pi) + i(\sqrt{3}\ln 3 - \pi - 2n\pi)}$

$= \frac{1}{3} e^{-\sqrt{3}(\pi+2n\pi) + i(\sqrt{3}\ln 3 - \pi)}$

$= \frac{-1}{3} e^{-\sqrt{3}(2n+1)\pi + i(\sqrt{3}\ln 3)}$

Principle value of $(-3)^{-1+\sqrt{3}i}$
$= \frac{-1}{3} e^{-\sqrt{3}\pi + i(\sqrt{3}\ln 3)}$

Trigonometric function

Defn: $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

In terms of xy-coordinate,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$= \frac{e^{ix-y} + e^{-ix+y}}{2}$$
$$= \frac{1}{2}\left(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right)$$
$$= \cos x \frac{e^{y}+e^{-y}}{2} + i \sin x \frac{e^{-y} - e^{y}}{2}$$
$$= \cos x \cosh y - i \sin x \sinh y.$$

$$\Rightarrow |\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$
$$= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y$$
$$= \cos^2 x + \sinh^2 y.$$

Inverse of Trigonometry function

Example: 1) Find $\cos^{-1} z$.

Ans: Let $w = \cos^{-1} z$.

Then $\cos w = z$

$$\Rightarrow \frac{e^{iw} + e^{-iw}}{2} = z$$

$$\Rightarrow (e^{iw})^2 - 2z(e^{iw}) + 1 = 0$$

$$\Rightarrow \quad e^{iw} = \frac{2z + \sqrt{4z^2-4}}{2}$$ ← multi-valued

$$= z + \sqrt{z^2-1}$$

$$= z + i\sqrt{1-z^2}$$

$$\Rightarrow \quad iw = \log\left(z + i\sqrt{1-z^2}\right)$$

$$\Rightarrow \quad w = -i\log\left(z + i\sqrt{1-z^2}\right)$$

2) Find $\cos^{-1} i$.

<u>Ans</u>: $\cos^{-1} i = -i\log\left(i + i\sqrt{1-i^2}\right)$

$= -i\log\left(i + \sqrt{2}\, i\right)$ or $-i\log\left(i - \sqrt{2}\, i\right)$

$= -i\log\left((1+\sqrt{2})i\right)$ or $-i\log\left((1-\sqrt{2})i\right)$

$= -i\left(\ln(1+\sqrt{2}) + i\left(\frac{\pi}{2}+2n\pi\right)\right)$ or $-i\left(\ln|1-\sqrt{2}| + i\left(-\frac{\pi}{2}+2n\pi\right)\right)$

$= \left(\frac{\pi}{2}+2n\pi\right) - i\ln(1+\sqrt{2})$ or $\left(-\frac{\pi}{2}+2n\pi\right) - i\ln(\sqrt{2}-1)$